Pseudo-Anosov diffeomorphisms in new smooth structures

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There are diffeomorphisms on a compact surface S with uniformly hyperbolic 1 dimensional stable and unstable foliations if and only if S is a torus: the Anosov diffeomorphisms. So, what is happening on the other surfaces? This question leads us to the study of pseudo-Anosov maps. Both Anosov and pseudo-Anosov maps appear as periodic points of the geodesic Teichmuller flow T_t on the unitary tangent bundle of the Moduli space over S. We observe that the points of pseudo-Anosov maps are regular (the foliations are like the ones for the Anosov automorphisms) except for a finite set of points, called singularities, that are characterized by their number of prongs k. The stable and unstable foliations near the singularities are determined by the real and the imaginary parts of the quadratic differential $\sqrt{z^{k-2}(dz)^2}$. By a coordinate change $u(z) = z^{k/2}$ the quadratic differential $z^{k-2}(dz)^2$ gives rise to the quadratic differential $(du)^2$ and, in this new coordinates, the pseudo-Anosov maps are uniform contractions and expansions of the stable and unstable foliations. We use this fact, to inspire us, to construct new pseudo-smooth structures near of the singularities such that the pseudo-Anosov maps are smooth, in this pseudo-smooth structures, and have the property that the stable and unstable foliations are uniformly contracted and expanded by the pseudo-Anosov dynamics, as we pass to describe briefly. Let $\mathbb{H}$ be the set of points $(x, y) \in \mathbb{R}^2$, with $y \geq 0$, and let T_k be the union $\sqcup_{j\in\mathbb{Z}_k}\mathbb{H}_{j\pi}$, where $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$. We obtain the paper surface Σ_k by identifying in T_k the points $(x, 0) \in \mathbb{H}_{(j+1)\pi}$ with $(-x, 0) \in \mathbb{H}_{j\pi}$. The sets Σ_k have an intuitive notion of straight lines and, so, of vectors. This allow us to construct a pseudo-linear algebra, where the sum of vectors with origin at the singularity is not well-defined for all vectors, but just for vectors forming an angle smaller than π. The product of a vector v and a constant λ is defined for positive constants. We define pseudo-linear spaces which give us natural splittings at the singularity. We construct a pseudo-linear algebra which is the first step to construct the notion of the derivative of a map at a singularity. Like that, we get a pseudo-smooth structure at the singularity, leading us to construct pseudo-smooth manifolds, pseudo-smooth submanifolds, pseudo-smooth splittings and pseudo-smooth diffeomorphisms. We prove the Stable Manifold Theorem (A. Pinto), in such pseudo-smooth manifolds, that gives pseudo-smooth pseudo-Anosov diffeomorphisms. We recover the duality given by Mañé-Bochi Theorem in the torus to the other surfaces (M. Bessa-A. Pinto-M. Viana): Let S be a pseudo-smooth manifold with a pseudo-volume form ω. There is a residual set $\mathcal{R}$ contained in the set of all C^1 pseudo-smooth diffeomorphisms, preserving the volume form, such that if $f \in \mathcal{R}$ then either f is a C^1 pseudo-smooth pseudo-Anosov diffeomorphism or has almost everywhere both Lyapunov exponents zero.